Angular Velocity, Center of Mass, Center of Gravity

Angular Velocity $\vec{\omega}$

Angular velocity also referred to as rotational velocity is expressed as

$$\omega = \frac{d\theta}{dt}$$

$d\theta$ is the change in angular rotation. $\vec{\omega}$ can be express in term of linear velocity as

$$\vec{v} = \vec{\omega} \text{ x } \vec{r}$$

where $\vec{v}$ is the linear velocity and $\vec{r}$ is the position vector of the particle.

Example:

A rigid body is rotating about an axis through the point $(3, -1, -2)$. If particle at the point $(4, 1, 0)$ has velocity $4\hat{i} - 4\hat{j} + 2\hat{k}$, and at the point $(3, 2, 1)$ has velocity $6\hat{i} - 4\hat{j} + 4\hat{k}$. Find the magnitude and direction of the angular velocity of the body.

Solution:

Let A, B, C are $(3, -1, -2)$, $(4, 1, 0)$, $(3, 2, 1)$ respectively.

$$\vec{AB} = \hat{i} + 2\hat{j} + 2\hat{k}, \;\; \vec{AC} = 3\hat{j} + 3\hat{k}$$

$$\vec{v_1} = (4\hat{i} - 4\hat{j} + 2\hat{k}), \;\; \vec{v_2} = (6\hat{i} - 4\hat{j} + 4\hat{k})$$

$$\vec{v_1} = \vec{\omega} \times \vec{AB}, \;\; \vec{v_2} = \vec{v_2} = \vec{\omega}_2 \times \vec{AC}$$

$$\vec{w}_2 - \vec{w}_3 = 2 \tag{1}$$

$$2\vec{w}_1 - \vec{w}_3 = 4 \tag{2}$$

$$2\vec{w}_1 - \vec{w}_2 = 2 \tag{3}$$

$$\vec{w}_1 = \frac{4}{3} \tag{4}$$

$$\vec{w}_2 = \frac{2}{3}, \ \ \vec{w}_3 = -\frac{4}{3}$$

$$\vec{w} = \vec{w}_1 + \vec{w}_2 + \vec{w}_3$$

$$= \frac{4}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{4}{3}\hat{k}$$

$$|\vec{w}| = 2$$

Direction of the angular velocity of the body:

$$\frac{\vec{w}}{|\vec{w}|} = (\frac{2}{3}, \ \frac{1}{3}, \ -\frac{2}{3})$$

Center of Mass (COM)

Center of mass is the point at which the whole mass of the body is assumed to be concentrated. The mass distribution around COM is uniform.

For two particles separated by a distance d where the origin is chosen as at the position of particle

$$X_{COM} = \frac{m_2}{m_1 + m_2} d$$

For two particles, for an arbitrary choice of origin:

$$X_{COM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

For many particles, we can generalize the equation where $m = m_1 + m_2 + m_3 + ... + m_n$:

$$X_{COM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n}{M} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

In three dimension, we find the center of mass along each axis separately

$$X_{COM} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i, \;\; Y_{COM} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i, \;\; Z_{COM} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i$$

The center of mass is in the same direction regardless of the coordinate system used. It is the property of the particle, not the coordinates. More concisely, we can write in terms of vectors

$$\vec{r}_{COM} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i$$

M is the mass of the body $\vec{r}_i$ is the position vector for each particle.
Center of Gravity (COG):
The center of gravity is the point at which the whole weight of the body is supposed to be acting. The weight distribution of the body around COG is uniform.